FUNCTIONS: DOMAIN and RANGE

[Note: In this worksheet, "x" will refer to the *independent* variable of a function and "y" will refer to the *dependent* variable. In reality, any letter could stand for either variable.]

Mathematical functions operate a little like computers – you put something in at one end (a value for the "independent" variable) and you get something out at the other (a value for the "dependent" variable). The restriction for something to be a function is that the input value never yields more than one output value.

For example, for the function y = f(x) = x + 2, if I pick a value of 3 for x, then y = 5, and I get the ordered pair (3, 5). For any x value I put in, I get only one y value out – y is a function of x.

The terms "domain" and "range" both refer to the sets of values that are *possible* for a function's variables to have. Domain is the set of possible values of the independent variable and range is the set of possible values of the dependent variable. What does "possible" mean? For domain, it means real numbers that yield real numbers as function output. In real-world problems, domain values also have to make sense – you can't have a negative area or length or weight. Range is a little trickier – we'll get back to that.

For the function y = f(x) = x + 2, x can be any value, from negative infinity to positive infinity, and we would say the domain of this function is all real numbers ($\mathbb{R}$), or $(-\infty,\infty)$ in interval notation, or $\{x \mid -\infty < x < \infty\}$ in set builder notation (read as "all x such that x is between negative infinity and positive infinity"). All real numbers make sense and yield real results.

So, when can x not be just any value? Here is THE question you have to answer in order to define domain: **are there any x values I *must exclude* for some reason**?

Whatever gets excluded becomes in effect the definition of the domain for that function.

If I can't use a value of 5 for x, but I can use all other values, then $x \neq 5$ defines the domain.

If x has to be greater than 0, then $x > 0$ defines the domain.

Now here's the easy part – there are only two questions you have to ask in order to define domain:

#1 **Rational functions**: are there any x values that will result in a division by zero and therefore make the function undefined?

#2 **Radicals**: are there any x values that will result in a negative number under an even root radical sign (square root, fourth root, sixth root, etc.), yielding a non-real output?

If the answer is "no" to both these questions for a particular function, and there are no real-world constraints, then the domain is all real numbers – done!

Question #1 can be answered "yes" *only if* you have a rational function – if you don't, no problem with this one. Let's look at some rational functions.

Define the domain of $f(x) = \dfrac{1}{x - 7}$

Step 1. Rational function? – yes → denominator cannot equal zero

Step 2. Grab the denominator, set it equal to zero, and solve for x.

$x - 7 = 0 \rightarrow x = 7 \rightarrow$ this is what x cannot be

Step 3. Define the domain: $x \neq 7$ *or* $(-\infty, 7) \cup (7, \infty)$ *or* $\{x \mid x \neq 7\}$

Define the domain of $f(x) = \dfrac{x + 4}{x^2 - x - 6}$

Step 1. Rational function? – yes → denominator cannot equal zero

Step 2. Grab the denominator, set it equal to zero, and solve for x. Notice that, as long as the numerator doesn't have an even radical in it, we don't care about the numerator! We're only looking for x values that create a division by zero.

$x^2 - x - 6 = 0 \rightarrow (x - 3)(x + 2) = 0 \rightarrow x = 3$ or $-2 \rightarrow$ x cannot be these values

Step 3. Define the domain: $x \neq 3$ and $x \neq -2$ *or* $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$

Graphically, this type of exclusion from the domain is usually where there are vertical asymptotes. Division by zero results in no y value output (undefined); the graph just goes to infinity or negative infinity on either side of the asymptote. It's also possible to have a hole in the graph instead of an asymptote, but the function is still undefined at the hole and the corresponding x-value must still be excluded.

Define the domain of $f(x) = \dfrac{1}{x^2 + 1}$

Step 1. Rational function? – yes → denominator cannot equal zero

Step 2. Grab the denominator, set it equal to zero, and solve for x.

$x^2 + 1 = 0 \rightarrow x^2 = -1 \rightarrow x = \pm\sqrt{-1} \rightarrow$ wait a minute!

These solutions are imaginary numbers, and we said the domain must be real numbers. Let's look at that denominator again: $x^2 + 1$. In the real realm, this will never be zero. So there is no x value we cannot use.

Step 3. Define the domain: $(-\infty, \infty)$ or $\mathbb{R}$ (all real numbers)

Question #2 requires the presence of an <u>even root</u> term anywhere in the function – a square root, fourth root, sixth root, etc. The magnitude of the root is irrelevant – if it's even, the radicand (what's inside the radical sign) can't be negative.

Define the domain of $f(x) = \sqrt{x - 7}$

Step 1. Even root? – yes $\rightarrow$ radicand cannot be negative, or must be ≥ 0

Step 2. Grab the radicand, set it ≥ 0, and solve for x.

$x - 7 \geq 0 \quad \rightarrow \quad x \geq 7 \quad \rightarrow$ this is what x <u>must</u> be

Step 3. Define the domain: $x \geq 7$ *or* $[7, \infty)$ *or* $\{x \mid x \geq 7\}$

Define the domain of $f(x) = \sqrt[4]{x^2 - x - 6}$

Step 1. Even root? – yes $\rightarrow$ radicand cannot be negative, or must be ≥ 0

Step 2. Grab the radicand, set it ≥ 0, and solve for x.

$x^2 - x - 6 \geq 0 \quad \rightarrow \quad (x - 3)(x + 2) \geq 0 \quad \rightarrow$ solve the inequality by using a number line or test values

$\rightarrow \quad x \leq -2$ or $x \geq 3$

Step 3. Define the domain: $x \leq -2$ or $x \geq 3$ *or* $(-\infty, -2] \cup [3, \infty)$

Define the domain of $f(x) = \dfrac{1}{\sqrt{x - 7}}$

Step 1. Even root? – yes $\rightarrow$ radicand cannot be negative, or must be ≥ 0
Also: Rational function? – yes $\rightarrow$ denominator cannot *equal* zero

Now, since the radical in the denominator cannot equal zero, that puts an *additional* restriction on the radicand $\rightarrow$ the radicand must be > 0.

Step 2. Grab the radicand, set it > 0, and solve for x.

$x - 7 > 0 \quad \rightarrow \quad x > 7$

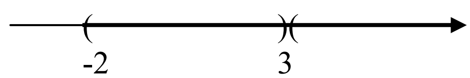
Step 3. Define the domain: $x > 7$ *or* $(7, \infty)$ *or* $\{x \mid x > 7\}$

Define the domain of f(x) = $\dfrac{\sqrt{x+2}}{x^2 - x - 6}$

Step 1. Even root? – yes → radicand cannot be negative, or must be ≥ 0
Also: Rational function? – yes → denominator cannot equal zero

Step 2. Now this gets a little complicated.
Constraint on the <u>numerator</u>: $x \geq -2$.
Constraint on the <u>denominator</u>: $x \neq -2$ and $x \neq 3$.

We have to combine the multiple restrictions so that all of them are met. Drawing a number line picture can help.
We have to start at -2 because of the numerator constraint and go to the right.
We can't include -2 because of the denominator constraint, and we can't include 3.



Step 3. Define the domain: $x > -2$ and $x \neq 3$ *or* $(-2, 3) \cup (3, \infty)$

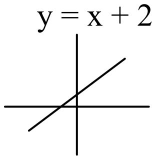
Range (y values)

To determine the range, you have to look at the nature of the function to see if there are any restrictions on the y values. Can y assume any value, or does it have a maximum or a minimum value? Restrictions on the domain may restrict or not restrict the range. The domain may be restricted but the range is not. Or, the domain may be all real numbers but the range is restricted. *It's extremely helpful to have some idea how the function looks when graphed.*

Let's look at the possible combinations.

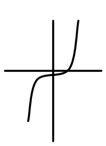
1. Domain unrestricted, range unrestricted

$y = x + 2$



For a straight line function, x can be any value and y can be any value.
Domain: $(-\infty, \infty)$ or $\mathbb{R}$
Range: $(-\infty, \infty)$ or $\mathbb{R}$

$y = x^3$

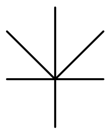


For this cubic function, x can be any value, and, because of the odd-number exponent, y can also be either negative or positive with no restriction
Domain: $(-\infty, \infty)$ or $\mathbb{R}$
Range: $(-\infty, \infty)$ or $\mathbb{R}$

2. Domain unrestricted, range restricted

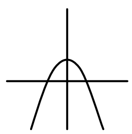
$y = |x|$

In an absolute value function, x can be any value, but y will never be negative because of the absolute value
Domain: $(-\infty, \infty)$ or $\mathbb{R}$
Range: $y \geq 0$ or $[0, \infty)$



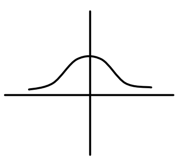
$y = -x^2 + 1$

This function will graph as a downward-opening parabola with a maximum value of positive 1. For quadratics, you can find the minimum or maximum y value by finding the vertex of the parabola.
Domain: $(-\infty, \infty)$ or $\mathbb{R}$
Range: $y \leq 1$ or $(-\infty, 1]$



$y = \dfrac{1}{x^2 + 1}$

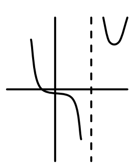
As we found earlier, even though this is a rational function, the domain is unrestricted. What about the range? Without graphing it, we can determine that when x is zero, y is one. If x is greater than or less than zero, the denominator is greater than one and y is therefore less than one. So 1 is the maximum value of y. In addition, y will always be positive (or > 0).
Domain: $(-\infty, \infty)$ or $\mathbb{R}$
Range: $0 < y \leq 1$ or $(0, 1]$



3. Domain restricted, range unrestricted

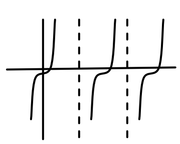
$y = \dfrac{x^3}{x - 2}$

This function has a vertical asymptote at $x = 2$. The graph goes to negative infinity on the left side of the asymptote, but makes an "s-curve" further to the left and curves up to positive infinity. So all y values are possible.
Domain: $x \neq 2$ or $(-\infty, 2) \cup (2, \infty)$
Range: $(-\infty, \infty)$ or $\mathbb{R}$



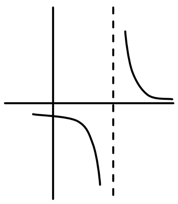
$y = \tan x$

This function has recurring vertical asymptotes at every odd multiple of $\pi/2$, where $\cos x = 0$. Between the asymptotes, the function values go from negative infinity to positive infinity.
Domain: $x \neq \pi/2 + n\pi$ where n = any integer
Range: $(-\infty, \infty)$ or $\mathbb{R}$



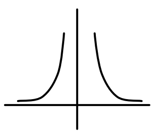
4. Domain restricted, range restricted

$y = \frac{1}{x-7}$ This rational function has a vertical asymptote at x = 7. To the left of the asymptote, the graph goes to negative infinity. To the right, the graph goes to positive infinity. Does that mean that the range is unrestricted? Check the rules for graphs of rational functions – there is also a horizontal asymptote at y = 0 (the x-axis), and y can never equal 0 because the numerator can never be zero. (does not cross the asymptote).



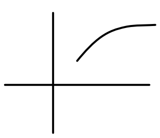
Domain: $x \neq 7$ or $(-\infty, 7) \cup (7, \infty)$
Range: $y \neq 0$ or $(-\infty, 0) \cup (0, \infty)$

$y = \frac{1}{x^2}$ In this rational function, there is a vertical asymptote when x = 0. Because x is squared, y will always be positive and go to positive infinity near the asymptote. Again, the x-axis is a horizontal asymptote which the graph does not cross.



Domain: $x \neq 0$ or $(-\infty, 0) \cup (0, \infty)$
Range: $y > 0$ or $(0, \infty)$

$y = 5 + \sqrt{x-3}$ x must be ≥ 3 because of the radical. Since a square root is considered positive unless otherwise stated, the smallest number the radical can be is zero, so the smallest number that y can be is 5 + 0 = 5.



Domain: $x \geq 3$ or $[3, \infty)$
Range: $y \geq 5$ or $[5, \infty)$